Assume P(k) true.

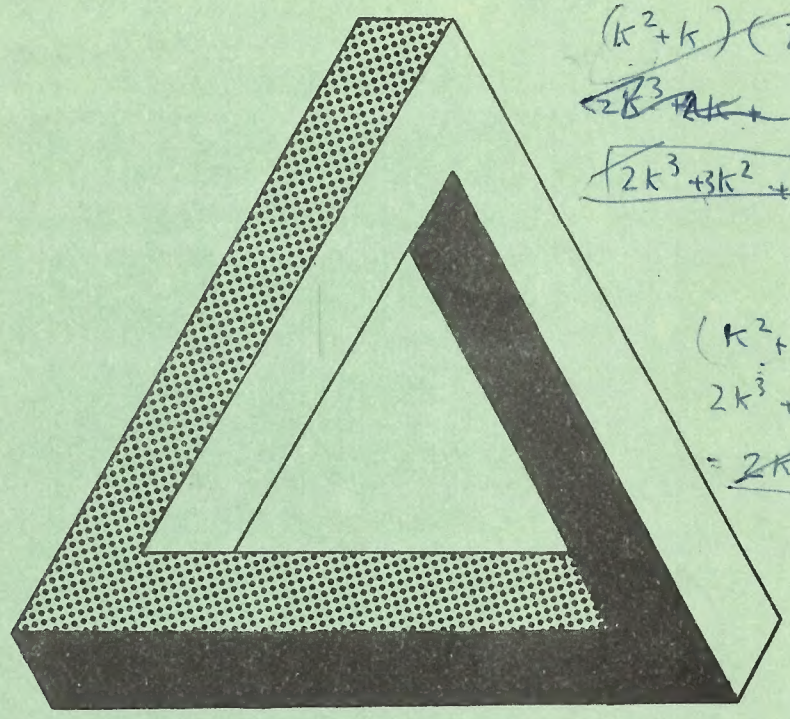
$\frac{k(k+1)(2k+1)}{6} = Q_k$ where Q is another

RTP P(k+1) true $\frac{(k+1)(k+2)(2k+3)}{6} = Q_{k+1}$

$2k^2 + k.$

$2k^2 + 3 + 5k.$

TRIGON



$(k^2+k)(2k+1)$

$\cancel{2k^3 + 2k^2 +}$ $2k^3 + k^2 + 2k^2 + k$

$\cancel{2k^3 + 3k^2 + k.}$

$(k^2 + \cancel{3k+3})(2k+3)$

$2k^3 + 3k^2 + 9k + 9 + 6k^2 + 6k.$

$= \cancel{2k^3} + 9k^2 + 15k + 9$

$Qk + 6k^2 + 14k + 9$

$\frac{(n)(n+1)(n+2)(n+3)}{6} + 1 = Q$

SCHOOL MATHEMATICS JOURNAL

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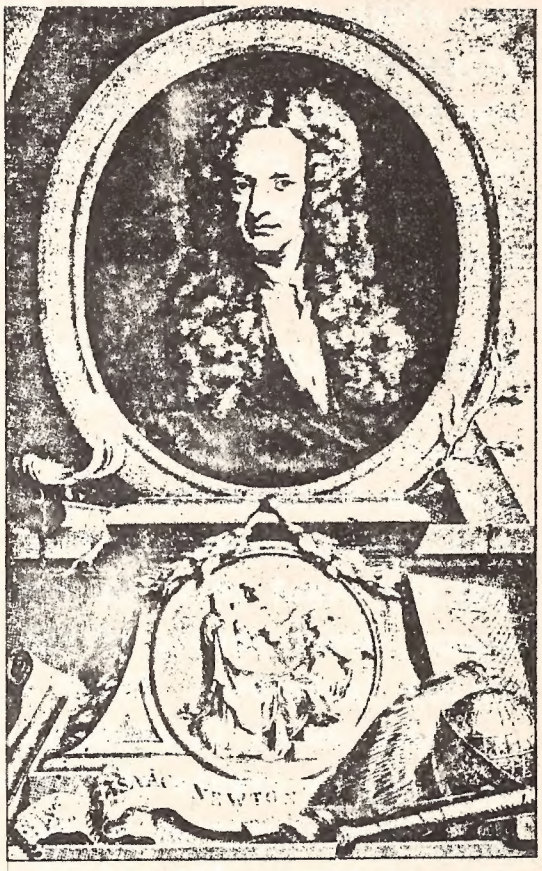
All material which is suitable for publication will be acknowledged in the Journal.

NEWTON - THE MATHEMATICAL GENIUS



"Nature and Nature's laws lay hid in night; God said, 'Let Newton be,' and all was light."

Such extravagant praise is understandable when we reflect that Isaac Newton is generally regarded as the greatest mathematician of all time. How odd then that such a unique man should consider that mathematics was not his chief concern. In fact, the three great loves in Newton's life were alchemy, chemistry, and theology; mathematics, physics, and astronomy were merely enjoyable diversions.

As you read about the life of this remarkable English inventor, mathematician, scientist, thinker, and university lecturer, you will begin to appreciate why another eminent mathematician, Leibniz, said of him: "Taking mathematics from the beginning of the world to the time when Newton lived, what he did was much the better half".

Newton's Early Life

The world is extremely fortunate that Isaac Newton was ever able to make his considerable impact upon history. Born as a premature baby on Christmas Day of 1642 (the year that the famous mathematician-scientist Galileo died), he was not expected to live long. Thankfully these predictions proved incorrect. At school he showed no promise until one fateful day when the class bully picked on him. Screwing up his courage, Isaac challenged the bully to a successful fight and then added to his triumph by regularly topping the class. Even at this age his originality showed itself in the clever devices he constructed. For example, he built an ingenious mechanical mouse-driven mill to grind wheat into flour, a

water-driven clock, and kites which carried lanterns into the night air, startling people who thought they were comets. After finishing school, Isaac was made to help on the farm where he was born. However, he was not the farming type. One day his uncle discovered him under a bush absorbed in a book on mathematics, so it was decided that the young thinker should return to school, and then go on to Cambridge University.

Newton's Rise to Greatness

Historians tend to divide Newton's career into three stages. The first third of his life was devoted to education and the formulation of his theories; the second third to the development, consolidation, and publication of those theories; and the final third to reaping the honours. At Cambridge Newton drank deeply of the many exciting ideas being taught by his lecturers. After reading widely and absorbing all that was then known in mathematics, he began to make new advances and even to create wholly new branches of mathematics. During 1665-66 an outbreak of the dreaded plague forced Newton to live at home. The result was one of the most productive periods of discovery in all mathematical history, for it was during this time that Newton crystallized many of his new findings. The most important of these were:

(i) the binomial theorem (the expansions of $(x + a)$, $(x + a)^2$, $(x + a)^3$, ...),

(ii) the calculus (which he referred to as "my method"),

(iii) the law of universal gravitation, and

(iv) the nature of light.

Any of these would have guaranteed him immortality; and bear in mind that Newton was only twenty-three years old! No wonder one of his teachers described him as an "unparalleled genius". Yet Newton's humble and sincere explanation was: "If I have seen a little farther than others, it is because I have stood on the shoulders of giants".

Newton's University Professorship

In 1669 Newton's ex-teacher resigned, so that his more brilliant student could take his place as Professor of Mathematics. Thus began an eighteen year program of lecturing and experimenting. Probably most of his time was spent in the hopeless search for the secret of transmuting base metals into gold, or in composing theological treatises. The rare occasions when he published his theories were rewarded by so much criticism and controversy that he vowed never to publish again.

During these years he continued to develop his theories on gravity. Legend has it that the principle came to him in a moment of inspiration after being hit on the head by a falling apple. Undoubtedly this did not happen, but it is interesting to reflect that the force which pulls an apple to the ground is precisely the same kind of force as that which keeps the moon in its orbit around the earth, and the planets in their orbits around the sun. This principle was stated as follows: "Any two particles in the universe attract one another with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them". By means of this principle Newton was able to explain Kepler's laws relating to the motions of the planets, the behaviour of the tides, the flattening of the poles, and the motions of planets, satellites and comets about the sun. "Newton's proof of this wonderful harmony - a mathematical harmony - in the universe revolutionized man's thinking... By showing that the same laws and forces operating in every man's backyard reach out and govern the most distant star, Newton fostered the suspicion that the heavenly bodies, like the earth, are less than divine".

Encouraged by his close friend, Halley (after whom the famous comet is named), Newton decided to publish once more, in the hope of a kinder reception by the scientific world. So remarkable were his powers of concentration that he would often forget about meals and dress, visitors and sleep. One

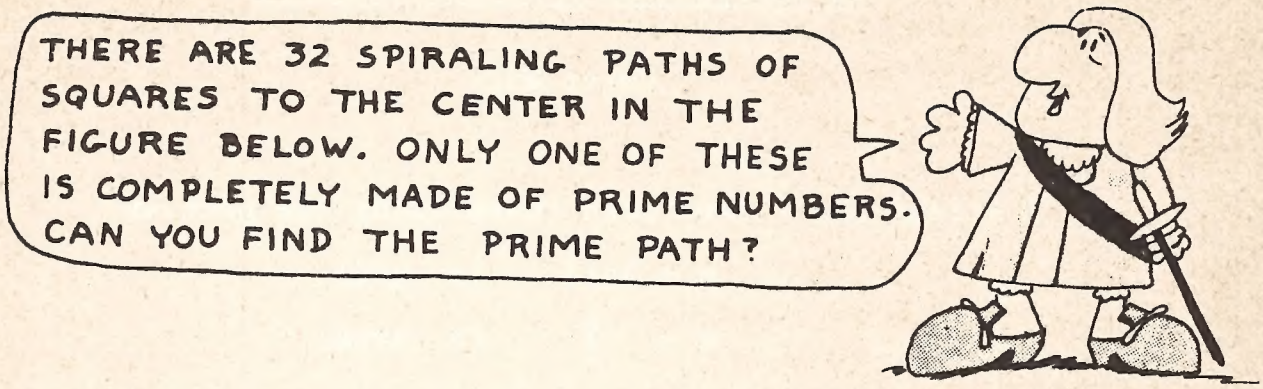
story tells how absent-minded he could be in one of these periods of intense concentration. Once when dinner was served to him he wandered off, deep in thought, leaving a visitor who was waiting to speak to him. Eventually the hungry visitor grew tired of waiting, and ate the entire dinner. Later, when Newton returned and saw the empty plate, he remarked, "How absent-minded we philosophers are. I really thought that I had not dined". In a remarkably short space of time he completed his magnum opus, Principia Mathematica, but at a considerable cost to his physical and mental health.

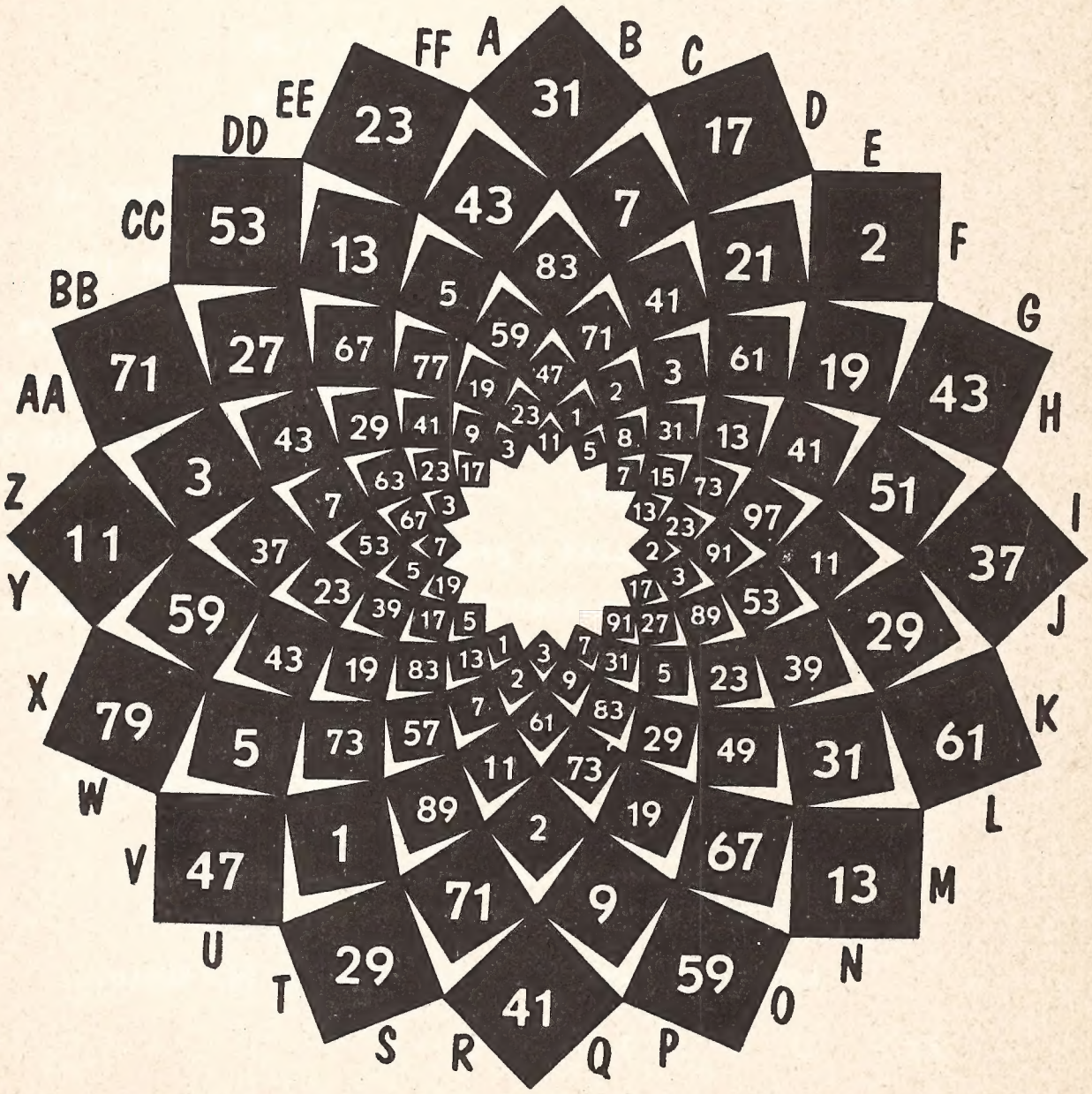
Newton's Career as a Master of the Mint

In the final third of Newton's life he took little interest in mathematics or science, apart from the occasional puzzle that nobody else could solve. In 1688 he was elected to Parliament, and in 1700 he left Cambridge for London to become Master of the Mint. Although his health failed him at times, and he became very moody on occasions, his mental powers did not diminish. The remainder of his life was spent pursuing his theological interests and working hard at the Mint. Finally, at the impressive age of eighty-four years, he died, probably of a heart attack. Thus ended the distinguished life of a humble genius, who thought of himself as "a boy playing on the seashore, and diverting myself in, now and then, finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered around me".

Taken from:
"New Approach to Mathematics"
F. A. Hoskin and K. A. Dan
The Jacaranda Press.

PRIME MAZE





REPRINTED FROM AFTERMATH 3

ON PROGRAMMING A NUMBER SEQUENCE

Peter Wild

A friend and I have been researching the problem

$$n \to 3n+1 \quad (n \text{ odd})$$

$$n \to \frac{n}{2} \quad (n \text{ even})$$

For example,

$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26$$
$$\to 13 \to 40 \to 20 \to 10 \to 5 \to 16$$
$$\to 8 \to 4 \to 2 \to 1 \ldots$$

This sequence is mentioned in the Research Corner of School Mathematics Journal, Vol.9, No.3, August 1971. Although we have accumulated much data, the conjecture that all such sequences beginning with a positive integer reach 1 after a finite number of steps is as far from proof (or disproof) as ever.

We have devised a programme to test the conjecture for $n \leq 1000$. Two theorems enable us to save time on this programme.

Let SEQ(n) be the (ordered) set of values n produces. The set is complete as soon as 1 appears.

Ex. SEQ(3) = {3, 10, 5, 16, 8, 4, 2, 1}

SEQ(1) = {1}

Let n_k denote the kth number in SEQ(n).

Ex. $3_2 = 10$, $n_1 = n$

Peter Wild is a student from Western Australia.

<u>Theorem 1</u> Let x be a positive integer greater than 2. If for every integer $n \in [2,x]$ there exists an n_p such that $n_p < n_1$, then for all integers $n \in [2,x]$, $1 \in SEQ(n)$.

We leave the proof as an exercise. However, assuming that the conjecture holds over the interval $[1,n-1]$, in testing for n we can now stop as soon as we obtain an $n_p < n$, instead of generating $SEQ(n)$ to 1.

In particular,

(a) we need not consider any even n.

For if $n_1 \equiv 0 \pmod 2$, then $n_2 = \frac{n_1}{2} < n$

(b) we need not consider any n for which $n \equiv 1 \pmod 4$

For if $n_1 \equiv 1 \pmod 4$, then $n_3 = \frac{3n_1+1}{4} < n$.

<u>Theorem 2</u> Let k be a positive integer. Then

$1 \in SEQ(2k+1)$ if and only if $1 \in SEQ(3k+2)$

<u>Proof</u>. Let $n_1 = 2k + 1$

Then $n_2 = 6k + 4$

$n_3 = 3k + 2$

i.e. $SEQ(2k+1) = \{2k+1, 6k+4, 3k+2, \ldots\}$

Hence $1 \in SEQ(2k+1)$ if and only if $1 \in SEQ(3k+2)$

We see then that there is no need to test integers n for which $n \equiv 2 \pmod 3$. Although this does not save as much time as Theorem 1, it is well worth incorporating in the programme.

We found that for $n \in [1,1000]$, the greatest number produced is 250504, first occurring in $SEQ(703)$. The number requiring the greatest number of steps to reach 1 is 871, requiring 178 steps.

PROBLEMS

Solutions to Junior Problems will be accepted only from students in third or lower secondary years. Solutions of problems in the Open Section will be accepted from students in any grade from Matriculation downwards. Some, but not all problems in the Open Section require more advanced knowledge than those in the Junior Section.

All solutions sent in must be the result of the solvers' unaided efforts.

Where a solution is obtained by the co-operation of two participants, the solution should be jointly signed by both authors. Our published solutions sometimes have to be abbreviated, but solutions sent in should be given fully.

Problems in this issue (Vol.13, No.3). We hope to publish these solutions and the names of those who sent in good solutions in Vol.14, No.1. Solutions should be in the hands of the Editor within one month of receiving the Journal.

IMPORTANT. The solution of each problem must be written on a separate sheet, but the sheets may be sent in one envelope. Each sheet should be signed by the author(s) giving full name (in block letters), name of school and grade. You may send in solutions to one problem only or to any number of problems.

Solutions should be sent to The Editor, TRIGON, Mathematics Department, University of Adelaide, Adelaide 5001.

(In some cases, Mathematical Associations outside South Australia have made alternative arrangements for the marking of solutions.)

$\frac{100}{d} = 4 = \frac{400}{9d} = 36$

$\frac{90}{d} = 18 \quad \frac{180}{2d} = 36$

$\frac{180}{2d} = \frac{900}{9d}$

JUNIOR PROBLEMS

15. If 100 and 90 are divided in turn by the same positive integer d, the remainder of the division is 4 in the first case and 18 in the second case. What is the divisor d?

16. Using straightedge and compass only (no geoliner!) construct an angle of $52\frac{1}{2}^0$.

17. In a circle of music lovers, 14 people play the piano or violin, 8 people are piano players, and 5 people play both instruments. Find the number of violin players. 6.

18. Prove that in an isosceles triangle the sum of the distances from any point on the base to the other two sides is a constant.

19. A bus route leads up a hill along a uniformly sloping path from terminus G at the bottom to terminus H at the top. A bus A leaves G at 8:12 a.m., and a bus B leaves H at 8:18 a.m. The two meet exactly midway. Bus B arrives at G and stays there for a few minutes before starting upwards. Bus A arrives at H, turns, and starts downwards immediately. The two buses meet again at 8:44 a.m. at a point M, the distance HM being twice the distance GM.

    Find the duration of the ascent and the descent, and find the resting time at G.

20. A fraction $\frac{a}{b}$ has the following three properties:
(i) $\frac{a}{b} = \frac{2}{7}$ (ii) $a+b$ is a two-digit number
(iii) $a+b$ is a perfect square. Find all such fractions.

21. Show that the area of a right-angled triangle with integer sides is an integer.

OPEN SECTION

015. Show that $\tan(5^0)$ and $\tan^2(5^0)$ are irrational.

016. Show that the area of a right-angled triangle with integer side lengths is divisible by 6.
(For example: 3: 4: 5 Area = 6.
7:24:25 Area = 84.
Hint: show that one of the two shorter sides is divisible by 4 and one by 3 (possibly the same side).)

017. Let z,w be complex numbers with conjugates $\bar{z},\bar{w}$.
(i) Find the locus L of the points z satisfying

$$z\bar{z} = z + \bar{z}$$

(ii) Find the locus M of the points w satisfying

$$\bar{z}w = 4$$

where z moves on L.

018. Solve the equation $3^x + 4^x = 5^x$.

019. Show that edges AD,BC of a tetrahedron $ABCD$ are mutually perpendicular if and only if

$$AB^2 + CD^2 = AC^2 + BD^2$$

020. Show that the even powers of $\sqrt{2}+1$ approximate very closely to integer values: thus the difference between $(\sqrt{2}+1)^{2n}$ and the nearest integer becomes as small as we please if $2n$ is sufficiently large. Find $(\sqrt{2}+1)^{10}$ to the nearest integer, and estimate the error.

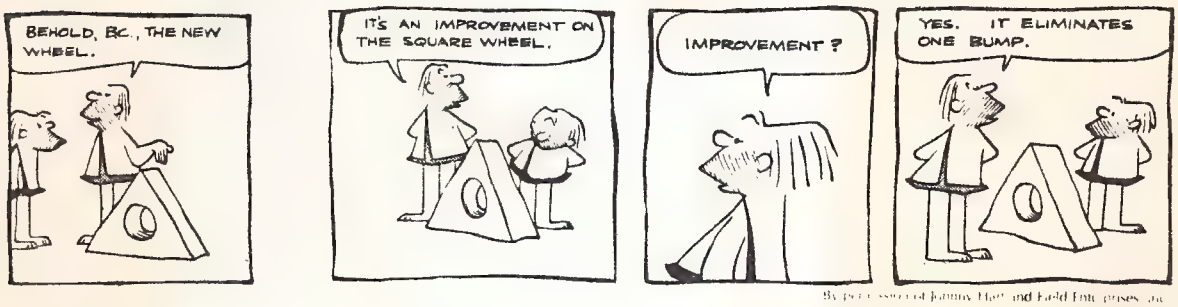
021. Show that it is possible to cut a hole of square cross-section in a wooden cube C, through which can be passed a wooden cube slightly larger than C.

MATHEMATICS IN NATURE

Devil's Post Pile

There are many families of geometric solids, one example being the regular prisms. A regular prism is a polyhedron that has a pair of congruent regular polygons for its bases and congruent rectangles for the rest of its faces. The bases of a prism are always parallel to each other and prisms are named according to their shape. For example, the prism with triangular bases is a triangular prism, the cube is a square prism, and so on.

One can easily find a multitude of examples of man-made regular prisms, but it is interesting to note that examples occur in nature as well. Several compounds (such as quartz) crystallize into regular prisms, but one of the most impressive displays of natural prisms is a set of tall columns of rock, some over 60 feet high, occurring in California and known as the Devil's Post Pile (pictured). Many of these columns have the shape of regular prisms, most hexagonal and others pentagonal. So mathematics occurs even in man's study of the earth!



A NOTE ABOUT ROOTS OF POLYNOMIALS WITH INTEGER COEFFICIENTS.

L. Bailey (Norwood High School)

A polynomial is an expression of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \ldots \qquad (1)$$

We will assume throughout that each polynomial we talk about has integer coefficients, that is $a_0, a_1, a_2, \ldots, a_n$ are all integers.

Now, it is not hard to show that $1+\sqrt{5}$ is a solution of the equation

$$p(x) = x^2 - 2x - 4 = 0$$

Suppose you are now asked to construct a polynomial which has for one of its roots $6 + 2\sqrt{5} = (1 + \sqrt{5})^2$. Here is an interesting solution to this problem. Write

$$p(x) = x^2 - 2x - 4 = (x^2 - 4) + (-2x)$$

Let

$$\begin{aligned} q(x) &= [(x^2-4) + (-2x)][(x^2-4) - (-2x)] \\ &= x^4 - 12x^2 + 16 \\ &= Q(x^2) \end{aligned}$$

where $Q(x) = x^2-12x+16$. Then $6 + 2\sqrt{5}$ is a solution of the equation $Q(x) = 0$.

In general suppose that α is a solution of the equation $p(x) = 0$ where $p(x)$ is as in (1); that is

$$p(\alpha) = 0. \tag{2}$$

Write

$$p(x) = p_1(x) + p_2(x) \tag{3}$$

where $p_1(x)$ contains the even powers of x in $p(x)$ and $p_2(x)$ the odd powers of x. Let

$$\begin{aligned} q(x) &= [p_1(x) + p_2(x)][p_1(x) - p_2(x)] \\ &= p_1(x)^2 - p_2(x)^2. \end{aligned} \tag{4}$$

Since $p_1(x)^2$, $p_2(x)^2$ contain only even powers of x so does $q(x)$, so we can write

$$q(x) = Q(x^2) \tag{5}$$

Then $Q(x)$ is the required polynomial. By equations (2) to (5)

$$\begin{aligned} Q(\alpha^2) = q(\alpha) &= [p_1(\alpha)+p_2(\alpha)][p_1(\alpha)-p_2(\alpha)] \\ &= p(\alpha)[p_1(\alpha) - p_2(\alpha)] \\ &= 0. \end{aligned}$$

SOLUTIONS TO PROBLEMS

IN VOL.13 NO.2

JUNIOR SECTION

8. Prove that

$$\frac{a^3+b^3}{2} > \left(\frac{a+b}{2}\right)^3$$

SOLUTION: It should first be noted that the problem 8 should have read: Prove that

$$\frac{a^3+b^3}{2} \geq \left(\frac{a+b}{2}\right)^3 ,$$

for $a,b > 0$. As it stands, we could set $a = -2$ and $b = 1$, and the inequality would not be true. Further if we set $a=b=1$ then the left hand side equals the right hand side. With this all in mind let us show that

$$M = \frac{a^3+b^3}{2} - \left(\frac{a+b}{2}\right)^3 \geq 0.$$

Expanding gives

$$M = \frac{a^3+b^3}{2} - \frac{a^3+3a^2b+3ab^2+b^3}{8}$$

$$\therefore \quad 8M = 4a^3+4b^3-a^3-3a^2b-3ab^2-b^3$$

$$= 3a^3+3b^3-3a^2b-3ab^2$$

$$\therefore \quad \frac{8}{3}M = a^3+b^3-a^2b-ab^2$$

$$= a^2(a-b)+b^2(b-a)$$

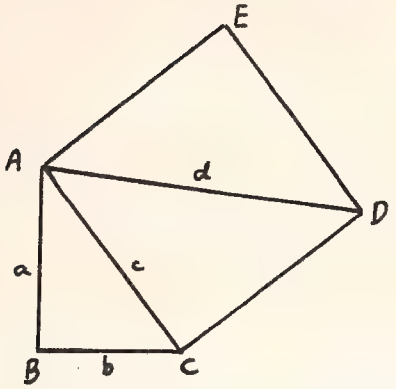
$$= (a^2-b^2)(a-b)$$

$$= (a+b)(a-b)(a-b)$$

$$= (a+b)(a-b)^2 \geq 0.$$

$\therefore$ $M \geq 0$ which proves the inequality.

9. Prove that the sum of the 'legs' of a right-angled triangle does not exceed the length of the diagonal of square on the hypotenuse.



SOLUTION:

Consider the diagram in which triangle ABC has a right angle at A and AEDC is a square on the side AC.
Let AB=a, BC=b, AC=c, AD=d.

The problem can be restated as: Show

$$a + b \leq d.$$

Let $P = a+b$, then $P^2 = (a+b)^2 = a^2+2ab+b^2$.

By Pythagoras' theorem we have $a^2+b^2 = c^2$

$$\therefore \quad P^2 = c^2 + 2ab.$$

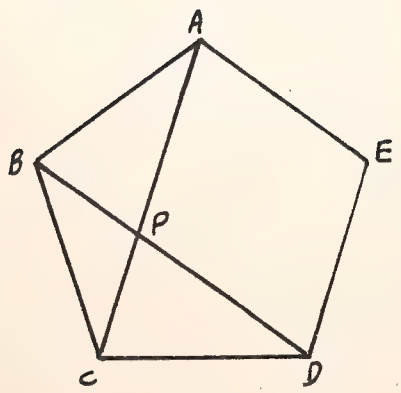
Now $$a^2 + b^2 - 2ab = (a - b)^2 \geq 0$$

$$\therefore \quad 2ab \leq a^2 + b^2 = c^2$$

$$\therefore \quad P^2 \leq c^2 + c^2 = d^2$$

and so $$P \leq d.$$

10. AC and BD are diagonals of a regular pentagon ABCDE. Show that the length of the longer segment of each diagonal is equal to the length of a side of the pentagon.



SOLUTION:

Let AC and BD intersect at P
We must show that PA=PD=AB.

Now $m\angle ABC = 108^0$ (angle in a pentagon)

$\therefore$ $m\angle BAC = 36^0$ (since ΔABC is isosceles)

$\therefore$ $m\angle CAE = 72^0$ (since $m\angle BAE = 108^0$)

In a similar way it can be shown that

$m\angle BDE = 72^0$ and hence $m\angle APD = 108^0$

$\therefore$ APDE is a parallelogram.

$\therefore$ AP = DE and PD = AE

But AE = ED = length of a side of the pentagon

$\therefore$ AP = PD = length of a side of the pentagon.

11. Show that $n(n+1)(2n+1)$ is exactly divisible by 6.

SOLUTION:

The product of 2 consecutive integers is divisible by 2. The product of 3 consecutive integers is divisible by 3 and hence by 6.

Now

$$\begin{aligned} n(n+1)(2n+1) &= n(n+1)(n+2+n-1) \\ &= n(n+1)(n+2) + (n-1)n(n+1). \end{aligned}$$

Since both terms on the right are divisible by 6, so is their sum.

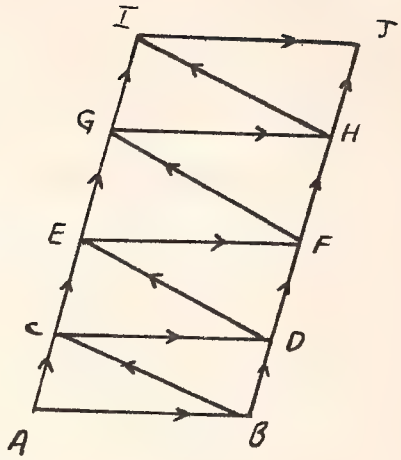
12. Any three digit number abc is divisible by 7 if and only if $2a+3b+c$ is divisible by 7. Why is this so? Can you generalize this to a rule for four or more digits?

SOLUTION:

$$\begin{aligned} abc_{10} &= 100a + 10b + c \\ &= (98a+7b) + (2a+3b+c) \\ &= 7(14a+b) + (2a+3b+c) \end{aligned}$$

Hence $(abc)_{10}$ is divisible by 7 if and only if $2a+3b+c$ is divisible by 7. This argument can be generalized to numbers other than 7 in precisely the same way.

13. ABCDEFGHIJ is a zig-zag track up a mountainside. At each labelled point other than I,J a steep alternative track branches off along AI or BJ. How many routes lead from A to J?



SOLUTION: Relabel the diagram as follows:

A=0, B=1, C=2, D=3, E=4, F=5, G=6, H=7, I=8, J=9.

Let U_n equal the number of paths leading from A=0 to n. (For example: $U_0=0$, $U_1=1$, $U_2=2$). Then

$U_n = U_{n-1} + U_{n-2}$ (the formula for the Fibonacci sequence!!)

We want U_9

$U_3 = U_2+U_1 = 3$; $U_4 = U_3+U_2 = 3+2 = 5$, $U_5 = U_4+U_3 = 5+3 = 8$

$U_6 = U_5+U_4 = 8+5 = 13$, $U_7 = U_6+U_5 = 13+8 = 21$, $U_8 = U_7+U_6 = 21+13 = 34$

$U_9 = U_8+U_7 = 34+21 = 55$.

14. A cardboard circle is cut along a diameter and a cone is formed out of one of the semicircular pieces. Now suppose that another cardboard circle is fitted to the base of the cone. This base circle is then cut along a diameter and a cone formed from one of the halves. What is the ratio of the volumes of the two cones?

SOLUTION: Because the two cones are constructed from similar pieces of cardboard (one the enlargement of the other), the two resulting cones will be similar. The slant height of cone 1 equals r (where r is the radius of the original piece of cardboard) and the slant height

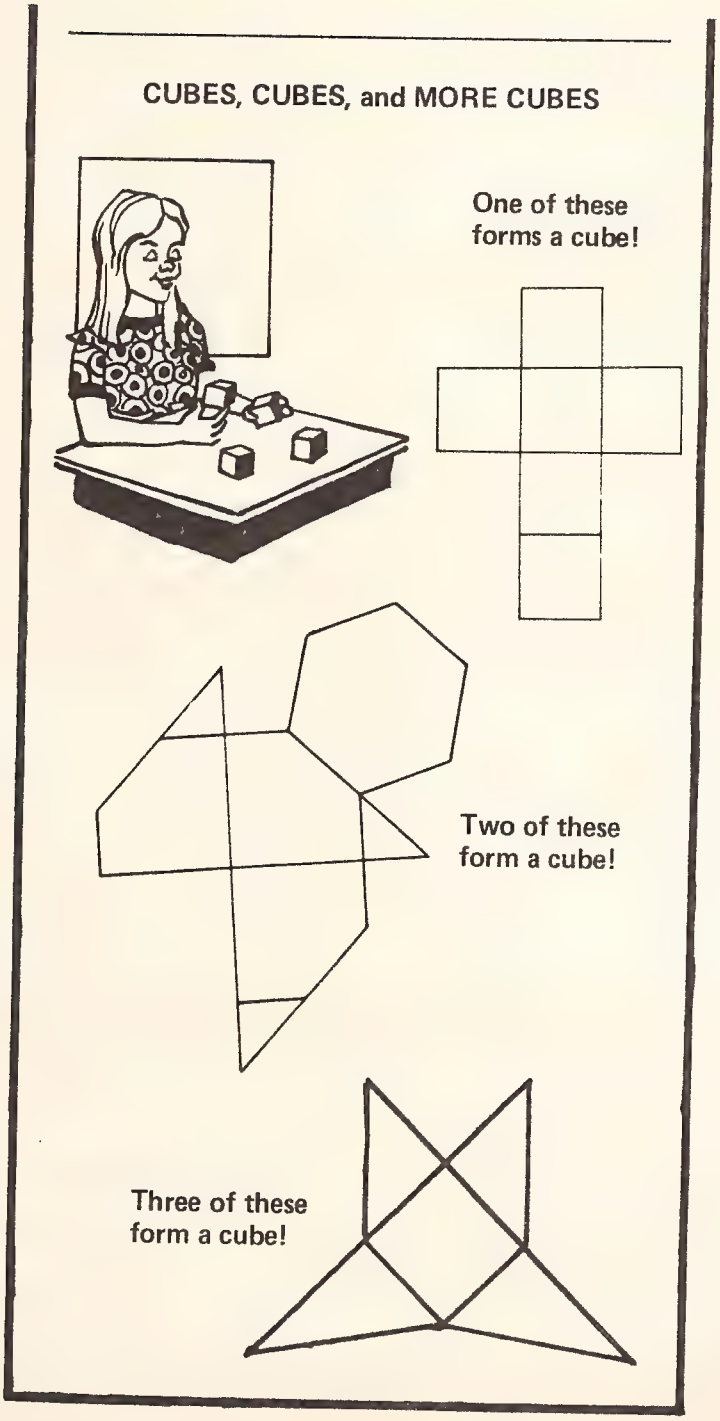
of cone 2 is $\frac{r}{2}$ (since the circumference of the base circle is the semicircular arc of radius r).

Hence Volume of Cone 1 : Volume Cone 2

$$= r^3 : (r/2)^3$$

$$= 1 : 1/8$$

$$= 8 : 1$$



OPEN SECTION

08. Show that by multiplying four consecutive numbers together and adding 1 we get a perfect square.

SOLUTION Denote by x the arithmetic mean of the four consecutive numbers $k-1$, k, $k+1$, $k+2$ (k an integer).

Then $$x = \frac{2k+1}{2},$$

and

$$\begin{aligned} P &= (k-1)k(k+1)(k+2) + 1 \\ &= \left(x - \frac{3}{2}\right)\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right)\left(x + \frac{3}{2}\right) + 1 \\ &= \left(x^2 - \frac{9}{4}\right)\left(x^2 - \frac{1}{4}\right) + 1 \\ &= x^4 - \frac{5}{2}x^2 + \frac{25}{16} \\ &= \left(x^2 - \frac{5}{4}\right)^2 \end{aligned}$$

Since $$x = \frac{2k+1}{2}, \quad x^2 - \frac{5}{4} = \frac{(2k+1)^2 - 5}{4} = k^2 + k,$$

an integer.

Thus P is the square of an integer.

09. If p, q, p_1, q_1 are real numbers, and

$$pp_1 = 2(q+q_1),$$

show that at least one of the equations

$$x^2 + px + q = 0$$

$$x^2 + p_1x + q_1 = 0$$

has two real roots.

SOLUTION Suppose that neither of the two equations has real roots. Then the discriminants of the equations can not be positive

i.e. $$p^2 \leq 4q$$

and $$p_1^2 \leq 4q_1$$

Since the geometric mean of two positive numbers is not

greater than the arithmetic mean, we have

$$pp_1 \le \frac{p^2+p_1{}^2}{2} \le 2(q+q_1)$$

If $p = p_1 = 4q = 4q_1$ we have equalities, and the two equations are actually identical, each having coincident roots.

Otherwise we obtain

$$pp_1 < 2(q+q_1)$$

contradicting the given condition.

Thus if the two equations are not identical, at least one of them must have two real roots.

010. Find all integer solutions of $x+y = xy$.

SOLUTION Write the equation in the form

$$y = xy-x = x(y-1).$$

This means that $y-1$ is a divisor of 1, so $y-1 = \pm 1$. Hence $y=0$ or $y=2$, and evaluating the corresponding x values we obtain $(x,y) = (2,2)$ or $(x,y) = (0,0)$.

011. Fourteen married couples live in a migrant hostel: five Italian couples, two Hungarian couples, three Yugoslav couples and four Greek couples. They decide to select a committee of seven from amongst themselves. In how many ways can this be done if each nationality is to be represented, and the committee is to have at least two women as members?

SOLUTION The problem is solved in three stages.

(a) Find the number N of ways a committee of 7 can be formed when each nationality is represented.

(b) Find the number N_0 of ways the committee can be formed with each nationality represented, but no women on the committee.

(c) Find the number N_1 of possible 4-nation committees with exactly 1 woman member.

The number of committees satisfying the requirements of the problem is then

$$C = N - N_0 - N_1$$

In each stage we must be careful to count each possibility once and only once. We use the "inclusion-exclusion" principle, illustrated in detail in (a).

(a) Let I,G,Y,H denote the sets of Italians, Greeks, Yugoslavs and Hungarians respectively. In order, these sets contain 10, 8, 6, 4 elements, a total of 28.

If no restriction is placed on the structure of the committee the total number of choices is

$$A = \binom{28}{7}$$

If no element of I is chosen, the committee does not satisfy the requirements. Thus of these A possibilities we must exclude

$$A_I = \binom{18}{7}$$

- the number of committees containing no Italian member. Similarly we must exclude

$$A_G = \binom{20}{7}, \quad A_Y = \binom{22}{7} \quad \text{and} \quad A_H = \binom{24}{7}$$

Now we must see that no exclusion is done twice. For example, the number of committees with no Italian or

Greek members is

$$A_{IG} = \binom{28-10-8}{7} = \binom{10}{7}$$

This was excluded twice, once as a subset of A_I, and once as a subset of A_G. Thus we must add $A_{IG} = \binom{10}{7}$ and similarly

$$A_{IY} = \binom{12}{7}, \quad A_{IH} = \binom{14}{7}, \quad A_{GY} = \binom{14}{7}, \quad A_{GH} = \binom{16}{7},$$

$$A_{YH} = \binom{18}{7}$$

Finally, repeating the argument, we must subtract again A_{IYH} (no Italians, Yugoslavs or Hungarians on the committee) and A_{GYH}

where

$$A_{IYH} = \binom{8}{7}, \quad A_{GYH} = \binom{10}{7}, \quad (A_{IGY} = 0 = A_{IGH})$$

Thus

$$N = \binom{28}{7}-\binom{18}{7}-\binom{20}{7}-\binom{22}{7}-\binom{24}{7}+\binom{10}{7}+\binom{12}{7}+2\binom{14}{7}$$

$$+ \binom{16}{7}+\binom{18}{7}-\binom{8}{7}-\binom{10}{7}$$

$$= 608960$$

(b) We repeat the above reasoning with the sets I_m, G_m, Y_m, H_m of Italian, Greek, Yugoslav and Hungarian mean. These sets have 5,4,3,2 elements respectively, and we obtain

$$N_0 = \binom{14}{7}-\binom{9}{7}-\binom{10}{7}-\binom{11}{7}-\binom{12}{7}+2\binom{7}{7}+\binom{8}{7}+\binom{9}{7}$$

$$= 2200$$

(c) We assume first that the only woman selected is Italian. She can be chosen in 5 different ways. For each of the choices, 6 men must be chosen out of 14 to make up the committee. The number of un-

restricted choices is

$$\binom{14}{6}$$

However, since the Italians are already represented by a woman, we need to exclude only those combinations where one or more of the <u>other</u> nationalities is missing. We apply the inclusion-exclusion principle as previously, and denote by B_I, B_G, B_Y and B_H the numbers of committes where the woman-representative is Italian, Greek, Yugoslav and Hungarian respectively. We obtain

$$B_I = 5\left\{\binom{14}{6}-\binom{10}{6}-\binom{11}{6}-\binom{12}{6}+\binom{7}{6}+\binom{8}{6}+\binom{9}{6}\right\}$$

$$B_G = 4\left\{\binom{14}{6}-\binom{9}{6}-\binom{11}{6}-\binom{12}{6}+\binom{6}{6}+\binom{7}{6}+\binom{9}{6}\right\}$$

$$B_Y = 3\left\{\binom{14}{6}-\binom{9}{6}-\binom{10}{6}-\binom{12}{6}+\binom{7}{6}+\binom{8}{6}\right\}$$

$$B_H = 2\left\{\binom{14}{6}-\binom{9}{6}-\binom{10}{6}-\binom{11}{6}+\binom{6}{6}+\binom{7}{6}\right\}$$

The total number of committees with 1 woman-representative is

$$N_1 = B_I + B_G + B_Y + B_H$$
$$= 24100$$

Thus $$C = 608960 - 2200 - 24100$$
$$= 582660$$

[The Editors wish to apologise for the inclusion of this long involved multinational problem!]

012. For what values of the common ratio can three successive terms of a geometric progression of positive numbers be sides of a triangle?

SOLUTION Assume that the side lengths are a,b,c where $a \le b \le c$, $b = ar$ and $c = ar$ $(r \ge 1)$.
We can construct a triangle with side lengths a,b,c if and only if the sum of the two smaller sides is greater than the greatest side:

$$a + b > c$$

$$\text{i.e.} \quad a + ar > ar^2$$

Hence $$r^2 - r - 1 < 0$$

Writing this as

$$(r-\tfrac{1}{2})^2 - \frac{5}{4} = \left(r - \tfrac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(r - \tfrac{1}{2} - \frac{\sqrt{5}}{2}\right) < 0$$

we obtain

$$\tfrac{1}{2} - \frac{\sqrt{5}}{2} < r < \tfrac{1}{2} + \frac{\sqrt{5}}{2} \qquad (*)$$

Since $r \ge 1$, we get

$$1 \le r < \tfrac{1}{2} + \frac{\sqrt{5}}{2}$$

By arranging the sides in descending order we similarly obtain

$$\tfrac{1}{2} - \frac{\sqrt{5}}{2} < r \le 1$$

Hence the inequalities (*) give the whole range of values for r.

013. Prove that the numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cannot be terms of a single arithmetic progression.

SOLUTION Suppose that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ belong to the same arithmetic progression in which the difference is d.
Then $$\sqrt{3} = \sqrt{2} + md$$

and $$\sqrt{5} = \sqrt{2} + nd \qquad (m,n \text{ integers})$$

Thus

$$d = \frac{\sqrt{3} - \sqrt{2}}{m} = \frac{\sqrt{5} - \sqrt{2}}{n}$$

or $$\frac{\sqrt{3} - \sqrt{2}}{\sqrt{5} - \sqrt{2}} = \frac{m}{n}$$

$$= r \text{ say, a rational number.}$$

Then
$$\sqrt{2} = \frac{r\sqrt{5} - \sqrt{3}}{r - 1}$$

Squaring,

$$2 = \frac{5r^2+3-2r\ 15}{(r - 1)^2}$$

and so

$$\sqrt{15} = \frac{2(r-1)-5r^2-3}{2r}$$

This is a contradiction since $\sqrt{15}$ is irrational. Thus our original assumption must be false, and $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cannot be terms of a single arithmetic progression.

014. (i) The equation of the curve H is

$$y = \frac{1}{x}.$$

Determine the shortest distance from the origin 0 to the curve.

(ii) The equation of the curve L is

$$(x^2+y^2)^2 = 4xy$$

Let OPQ be a ray from the origin intersecting the curve L in P and the curve H in Q. Find a relationship between lengths OP, OQ, and hence sketch the curve L.

<u>SOLUTION</u> (i) Use polar coordinates r,θ. Thus

$$x = r\cos\theta$$

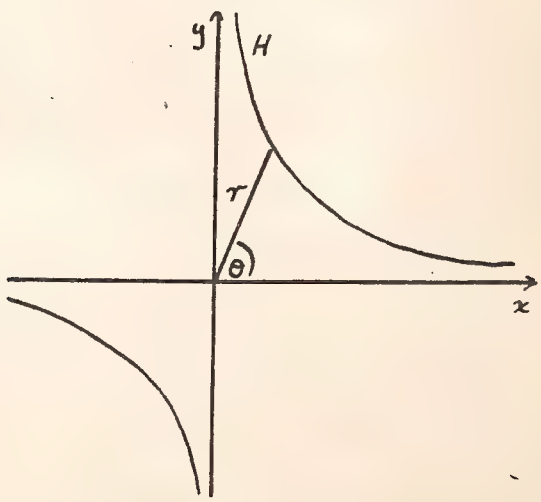
$$y = r\sin\theta$$

If (x,y) is on H, then

$$xy = r^2\sin\theta\cos\theta = 1$$

Thus

$$r^2 = \frac{1}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} \geq 2$$



Also, $r^2=2$ if and only if $\sin 2\theta=1$, whence

$$\theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \text{ and } r = \sqrt{2}$$

(ii) The equation of L in polar coordinates is

$$r^4 = 4r^2 \sin\theta \cos\theta$$

i.e. $$r^2 = 2 \sin 2\theta.$$

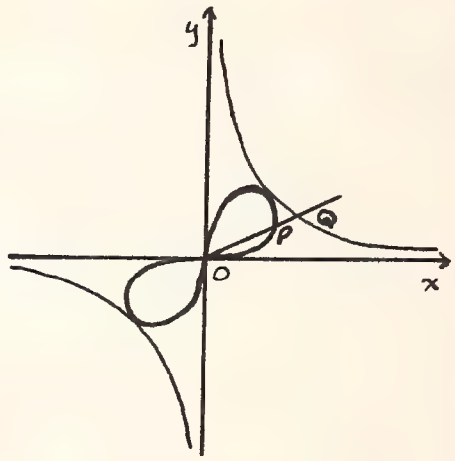
Notice in particular that $r \le \sqrt{2}$ here, equality occurring when $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

In general, if θ is the angle that OPQ makes with the x-axis, then for H

$$r_H{}^2 = \frac{2}{\sin 2\theta},$$

and for L

$$r_L{}^2 = 2 \sin 2\theta.$$



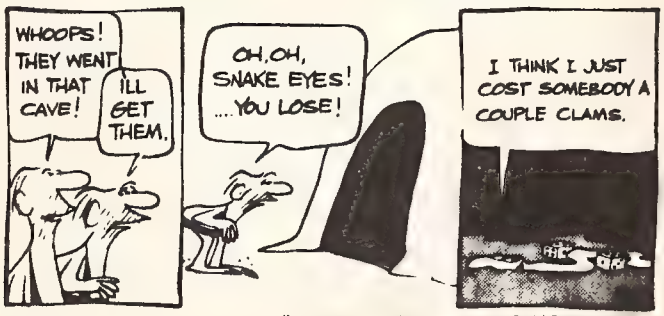
Hence for all such rays

$$r_H r_L = 2.$$

Thus if $\theta = \frac{\pi}{4}$, $r_H = r_L$,

and as $r_H \to \infty$, $r_L \to 0$.

L is a lemniscate, and is an inverse curve of the hyperbola H.



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Donovan Ell was killed on a lonely road two miles from Trenton at 3:30 a.m. on February 14. Five men were arrested a week later and questioned. Each man made four simple statements of which three were absolutely true and only one of them false. One of these five men killed Donovan.

Shorty: "I was in Chicago when Donovan was murdered. I never killed anyone. Red is the guilty man. Joey and I were pals."

Hank: "I did not kill Donovan. I never owned a revolver in my life. Red knows me. I was in Philadelphia the night of February 14."

Tony: "Hank lied when he said he never owned a revolver. The murder was committed on Valentines' Day. Shorty was in Chicago at that time when the murder was committed. One of us five is guilty."

Joe: "I did not kill Donovan. Red has never been in Trenton. I never saw Shorty before. Hank was in Philadelphia with me the night of February 14."

Red: "I did not kill Donovan. I have never been in Trenton. I never saw Hank before now. Shorty lied when he said I'm guilty."

WHICH ONE OF THE FIVE WAS THE MURDERER OF DONOVAN?

Red.

A shepherd was asked how many sheep he had in his flock. He replied that he could not say, but he knew if he counted them by twos, by threes, by fours, by fives, or by sixes, there was always one over, but if he counted them by seven there was none over. What is the smallest number that will answer the above conditions?

A man goes into a store and says to the owner: "Give me as much money as I have with me and I will spend $10." It is done and the man repeats the operation in a second and third store, after which he has no money left. How much did he start with?

	1	5	2				
	2	1	4	0	7		
	5	2	0	2	1		
	1	4	1	6	0		
				2	5	2	
				1	0	2	

Divide this diagram into four congruent parts, each containing six digits that add up to 14.

REPRINTED FROM THE MATHEMATICS STUDENT

HERON TRIANGLES

J. van der Hoek

Heron of Alexandria gave a formula for the area of a triangle in terms of its sides. If the lengths of its sides are a,b and c then the

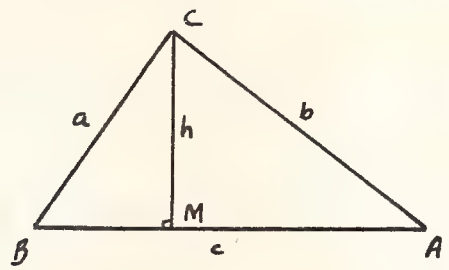
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \qquad (1)$$

where $2s = a+b+c$. If we set $a=13$, $b=14$, $c=15$ then we get Area = 84 (Check this out!)

A triangle whose sides and whose area are all integers is called a Heron triangle. Heron triangles have some remarkable properties!

Property 1 The angles in a Heron triangle have rational (fractional) values for their sine and cosine. (This means that 30^0, 45^0, or 60^0 could never be an angle of a Heron triangle (WHY?))

Proof: Consider the following Heron triangle.



Let its area be denoted by A. Then

$$A = \tfrac{1}{2}hc$$

and so

$$h = \frac{2A}{c} \qquad (2)$$

By Pythagoras' theorem,

$$AM^2 = CA^2 - CM^2 = b^2 - \left(\frac{2A}{c}\right)^2$$

$$\therefore \qquad AM = \sqrt{b^2 - \left(\frac{2A}{c}\right)^2} = \sqrt{Q}$$

where $Q = b^2 - \left(\frac{2A}{c}\right)^2$ which is a rational number. Now $\sqrt{Q}$ is rational or irrational depending on whether Q is a perfect square or not. Suppose if possible that $\sqrt{Q}$ is irrational, then by Pythagoras' theorem in triangle BCM,

$$a^2 = BC^2 = BM^2 + CM^2 = (c - \sqrt{Q})^2 + \left(\frac{2A}{c}\right)^2$$
$$= c^2 - 2\sqrt{Q}\, c + Q + 4A^2/c^2$$

from which it follows that

$$\sqrt{Q} = \{c^2 + Q + 4A^2/c^2 - a^2\}/2c \qquad (3)$$

But now the left hand side of (3) is irrational and the right hand side is rational, which is not possible. Hence $\sqrt{Q}$ must have been rational after all, which means that MA has rational length. From this it follows that $\sin \angle A = CM/CA$ and $\cos \angle A = MA/CA$ must both be rational.

Before giving another property about Heron triangles, let us first recall a few trigonometrical identities.

$$\tan \alpha/2 = \frac{1 - \cos \alpha}{\sin \alpha} \qquad (4)$$

$$\sin \alpha = \frac{2 \tan \alpha/2}{1+\tan^2\alpha/2} \qquad (5)$$

$$\tan(\alpha+\beta) = \frac{\tan \alpha + \tan \beta}{1-\tan \alpha \tan \beta} \qquad (6)$$

$$\tan(90^0-\alpha) = \cot \alpha = \frac{1}{\tan \alpha} \qquad (7)$$

$$\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C} = \lambda \qquad (8)$$

The last identity is the celebrated sine rule and λ is

some constant. (It equals the diameter of the circumcircle of the triangle ABC).

<u>Property 2</u> The area of a Heron triangle is divisible by 6.

<u>Proof</u>: In the triangle above let $m\underline{/A} = \alpha$, $m\underline{/B} = \beta$, $m\underline{/C} = \gamma$ for convenience.

Now $\tan \alpha/2$ and $\tan \beta/2$ are both rational numbers. This follows from property 1 and (4). Let $\tan \frac{\alpha}{2} = \frac{m}{n}$ and $\tan \frac{\beta}{2} = \frac{p}{q}$, where the two fractions are in <u>lowest terms</u> (m,n,p and q are all positive integers).

Then by identity (5)

$$\sin \alpha = \frac{2m/n}{1 + (m/n)^2} = \frac{2mn}{m^2 + n^2} \tag{9}$$

and

$$\sin \beta = \frac{2pq}{p^2 + q^2} \tag{10}$$

Now
$$\begin{aligned} \tan \frac{\gamma}{2} &= \tan \tfrac{1}{2}(180^0 - \alpha - \beta) \\ &= \tan\left(90^0 - \frac{\alpha+\beta}{2}\right) \\ &= \cot \frac{\alpha+\beta}{2} \quad \text{by (7)} \\ &= \frac{1}{\tan \frac{\alpha+\beta}{2}} \quad \text{by (7)} \end{aligned} \tag{11}$$

Now by (6),

$$\tan \frac{\alpha+\beta}{2} = \tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \frac{\tan \alpha/2 + \tan \beta/2}{1 - \tan \alpha/2 \tan \beta/2}$$

$$= \frac{\frac{m}{n} + \frac{p}{q}}{1 - \frac{m}{n} \cdot \frac{p}{q}}$$

$$= \frac{qm + qn}{nq - mp}$$

and so by (11)

$$\tan \frac{\gamma}{2} = \frac{nq - mp}{qm + pn}$$

and by (5)

$$\sin \gamma = \frac{2(nq-mp)}{qm+pn} \Big/ \left\{1 + \left(\frac{nq-mp}{qm+pn}\right)^2\right\}$$

$$= \frac{2(nq-pm)(np+mq)}{(m^2+n^2)(p^2+q^2)}$$

after some simplification!! Now by the sine rule (8)

$$a = \lambda \sin \alpha = \frac{2\lambda mn}{m^2+n^2}$$

$$b = \lambda \sin \beta = \frac{2\lambda pq}{p^2+q^2}$$

$$c = \lambda \sin \gamma = \frac{2\lambda(nq-pm)(np+mq)}{(m^2+n^2)(p^2+q^2)}$$

Since a,b,c are integers and as it can be shown that $(m^2+n^2)(p^2+q^2)$ has no factors in common with either n,m,p,q, $(nq-pm)$ or $(np+mq)$, 2λ must be a multiple of $(m^2+n^2)(p^2+q^2)$ so that this term will cancel into the numerator. Let

$$2\lambda = k(m^2+n^2)(p^2+q^2) \tag{12}$$

Then

$$a = kmn(p^2+q^2)$$

$$b = kpq(m^2+n^2)$$

$$c = k(nq-mp)(np+mq)$$

Now the area of triangle ABC

$$= \tfrac{1}{2}ab \sin \angle C$$

$$= \tfrac{1}{2}kmn(p^2+q^2)kpq(m^2+n^2).2.\frac{(nq-pm)(np+mq)}{(m^2+n^2)(p^2+q^2)}$$

$$= k^2mnpq(nq-pm)(np+mq)$$

Let us investigate the quantity

$$\Lambda = mnpq(nq-pm)(np+mq)$$

$$= mnpq\ XY \tag{13}$$

where $X = (nq-pm)$ and $Y = (np+mq)$. To prove property 2 we need only show that Λ is divisible by 6. We shall show separately that it is divisible by 2 and 3. If any of m,n,p or q is even then so is Λ. If all m,n,p,q are odd then nq is odd and pm is odd and so X is even which means that Λ is even. So far so good!
If any of m,n,p or q is divisible by 3 then so is Λ. Let us assume that none of m,n,p or q is divisible by 3. The first case would be true if m and n both gave the same remainder on division by 3, and the second case would hold if they gave different remainders on division by 3 (one remainder would be 1 and the other 2). A similar story holds for $p-q$ and $p+q$. Let us draw a table.

	m-n divisible by 3	m+n divisible by 3
p-q divisible by 3	3 divides X by (14)	3 divides Y by (16)
p+q divisible by 3	3 divides Y by (17)	3 divides X by (15)

$$X = (nq-pm) = (q-p)n + p(n-m) \tag{14}$$

$$= q(n+m) - m(p+q) \tag{15}$$

$$Y = (np+mq) = (p-q)n + q(n+m) \tag{16}$$

$$= p(n-m) + (p+q)m \tag{17}$$

In each case it follows that 3 divides either X or Y and so also Λ. Thus property 2 is true. Phew!

<u>Some examples</u>. We notice that k above is a common factor to a,b,c and the area so let us set $k=1$. We still then have a Heron triangle.

One warning when choosing n,m,q,q; makes sure that m,n have no factors in common, p,q have no factors in common and nq > np (otherwise c < 0).

m	n	p	q	a	b	c	Area = Λ
1	2	1	1	4	5	3	6
1	3	1	1	6	10	8	24
1	2	1	2	10	10	12	72
1	2	1	3	20	15	25	300
1	2	1	4	34	20	42	336

Some Problems to investigate

1. Find Heron triangles whose perimeters and areas are numerically equal [e.g. (6,8,10)]
2. Find Heron triangles whose sides are in an arithmetic progression; e.g. (3,4,5), (6,8,10).
3. Show that the perimeter of a Heron triangle is always even.
4. Investigate the possibility of "pasting" right angle triangles with integer sides together to obtain Heron triangles.
5. Show that any integer > 2 can be the side of a Heron triangle. In fact it can be the side of an infinite number of Heron triangles.

Reference: L.E. Dickson: "History of the Theory of Numbers" vol.2. Chelsea Publishing Co. N.Y. 1971 Reprint.

DIALOGUE - ANGELO'S PAGE

(Angelo Capozza is a matriculation student at Adelaide Boys' High School.)

Angelo: I believe that TRIGON would become more popular if students could submit problems and answers for publication. These problems could be in a different section altogether.

Editor: That sounds a good idea. We'll add 'problems' to the list on page 2!

Angelo: Here is a problem. Is it possible to write numbers using each of the ten digits 0,1,2,...,9 exactly once so that the sum of these numbers is 100?

Editor: I don't know.

Angelo: Let's try anything.

$$10 + 23 + 6 + 7 + 8 + 9 + 45 = 108$$

Editor: You're wrong.

Angelo: Let's try again.

$$19 + 28 + 30 + 7 + 6 + 5 + 4 = 99$$

Editor: That's close, but not good enough. Perhaps there is no way of obtaining 100.

Angelo: If there is a set of numbers whose sum is 100, then the numbers must have one or two digits, and no digit can appear more than once. Let t denote the sum of the digits representing tens. Now the sum of all the digits is

$$0 + 1 + 2 + \ldots + 9 = 45$$

so the sum of the 'units' digits is $45 - t$.

Hence we must solve

$$10t + (45 - t) = 100$$

This gives $$t = \frac{55}{9}$$

Editor: But that is ridiculous - t must be an integer.

Angelo: Right! So it is not possible to find the numbers in the problem. But if we insist that the sum be 153, then

$$10t + (45 - t) = 153$$

and so $$t = 12$$

Now we know that there is a solution!

Editor: I'm not so sure about that...

Angelo: I can prove it. Since $t = 12$, the sum of the tens is 120, and the sum of the units must be $45 - 12 = 33$. Now by trial and error

$$34 + 10 + 25 + 8 + 9 + 67 = 153$$

Editor: Yes, but what if you couldn't find a solution by trial and error? You might not always be able to find one. I wonder what the readers think? Actually Angelo, this is a very good problem because it raises further questions. For example, what are the smallest and largest numbers which can be obtained in the problem? And how many numbers in between can be obtained in this way?

(Readers with an answer (or a problem!) are invited to write to the Editor.)

7

The number seven seems to hold some fascination for man.

In the Book of Revelations there are seven stars and seven golden candlesticks which represent seven churches; a lamb with seven horns and seven eyes which are the seven spirits of God which are also seven lamps of fire; seven seals of the book; seven angles; seven trumpets; seven thunders; seven vials; seven kings and a dragon with seven heads.

Other religious uses of seven are the seven days of creation and the seven deadly sins. Matthew contains the story of the seven loaves to feed the four thousand with seven baskets of crumbs remaining. In Isaiah there are the seven gifts of the Holy Ghost.

Folklore contains the infamous man with seven wives met on the way to St Ives and the seven-league boots of Hop o' my Thumb. There is the dance of the seven veils, the seven pillars of wisdom, the seven sages of Greece, the seven Argive heroes, the seven wonders of the world and the constellation of stars known as the Seven Sisters.

Break a mirror and you will have not six, not eight but seven years bad luck. Why seven? Why the mystery or magic of seven?

That this attitude occurs so far back in history it is likely that it arises in the ancient world when priests, astrologers, the learned classes the world over were pre-occupied with observing and noting the motions of the sun, moon and stars. Is it any coincidence that the year was first divided into quarters of 91 days, thirteen weeks of

<u>seven</u> days?

Arithmetically seven occurs in some remarkable coincidences.

$$\frac{1}{7} = 0.142857142857142857\ldots = 0.\dot{1}4285\dot{7}$$

Considering the recurring digits 142857 as a six digit numeral we can form the following table:

1 x 142857 = 142857
3 x 142857 = 428571
2 x 142857 = 285714
6 x 142857 = 857142
4 x 142857 = 571428
5 x 142857 = 714285

It is interesting to note that

7 x 15873 = 111111
7 x 31746 = 222222
7 x 47619 = 333333
7 x 63492 = 444444
7 x 79365 = 555555
7 x 95328 = 666666
7 x 111111 = 777777
7 x 126984 = 888888
7 x <u>142857</u> = 999999

The number 142857 is unique in that it is the only number less than one million whose 'cyclic permutations' are multiples of itself. The next known number of this type is

052 631 578 947 368 421

A treat is in store for any person multiplying this number by 1, 2, 3, ... and setting the products down in rows as was done in the previous paragraph!

Mathematically or historically, seven has figured well.

MATCHSTICK GEOMETRY

Two definitions:

Remove: Pick up from the figure and do not return.

Move : Pick up from the figure and place in a different position in the figure.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

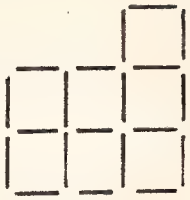


Fig. 5

1. In Figure 1, remove 2 matches to form 2 squares.
2. In Figure 1, move 2 matches to form 2 squares.
3. In Figure 2, move 3 matches to form 3 squares.
4. In Figure 2, move 4 matches to form 3 squares.
5. In Figure 3, remove 5 matches to form 3 congruent squares.
6. In Figure 3, remove 2 matches to form 4 squares.
7. In Figure 4, remove 5 matches to form 5 squares.
8. In Figure 4, remove 8 matches to form 2 squares.
9. In Figure 4, remove 8 matches to form 3 squares.
10. In Figure 4, remove 8 matches to form 4 squares.
11. In Figure 4, move 8 matches to form 3 squares.
12. In Figure 4, remove 6 matches to form 3 squares.
13. In Figure 4, remove 10 matches to form 2 squares.
14. In Figure 4, remove 12 matches to form 3 squares.
15. In Figure 5, move 7 matches to form 2 pairs of congruent squares.
16. With 12 matches, form a figure which is composed of 6 congruent regions.

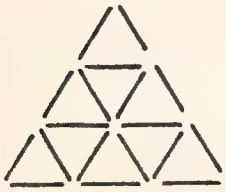


Fig. 6

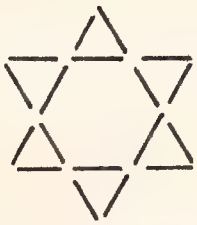


Fig 7

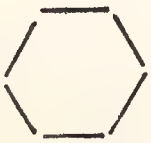


Fig. 8

Fig. 9

Fig. 10

17. In Figure 6, remove 5 matches to form 5 triangles.
18. In Figure 6, remove 6 matches to form 5 triangles.
19. In Figure 6, move 6 matches to form 6 congruent quadrilaterals.

20. In Figure 7, move 6 matches to form 6 congruent quadrilaterals.
21. In Figure 8, move 2 matches and add 1 match to form 2 rhombuses.
22. In Figure 9, move 4 matches to form 3 equilateral triangles.
23. In Figure 9, move 3 matches to form 6 congruent quadrilaterals.
24. In Figure 10, remove 3 matches to form 3 triangles.
25. In Figure 10, remove 4 matches to form 2 triangles, 1 trapezium and 1 parallelogram.

Fig. 11

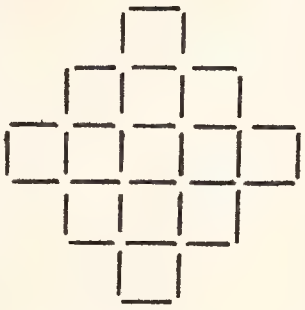


Fig. 12

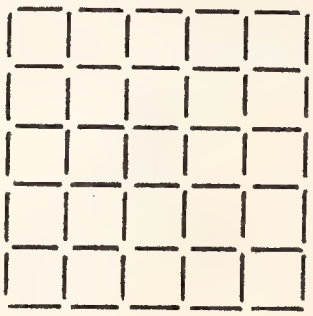


Fig. 13

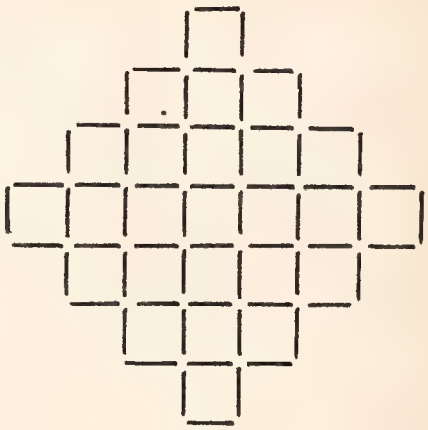
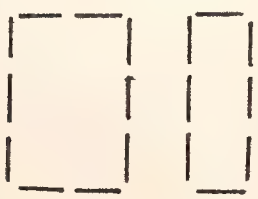


Fig 14

26. In Figure 11, remove 14 matches to form 6 congruent squares.
27. In Figure 12, remove 8 matches to form 6 squares.
28. In Figure 12, remove 4 matches to form 9 squares.
29. In Figure 12, remove 8 matches to form 9 congruent squares.
30. In Figure 13, remove 24 matches to form 9 congruent squares.
31. In Figure 14, remove 16 matches to form 12 congruent squares.
32. In Figure 14, remove 32 matches to form 6 squares.
33. 

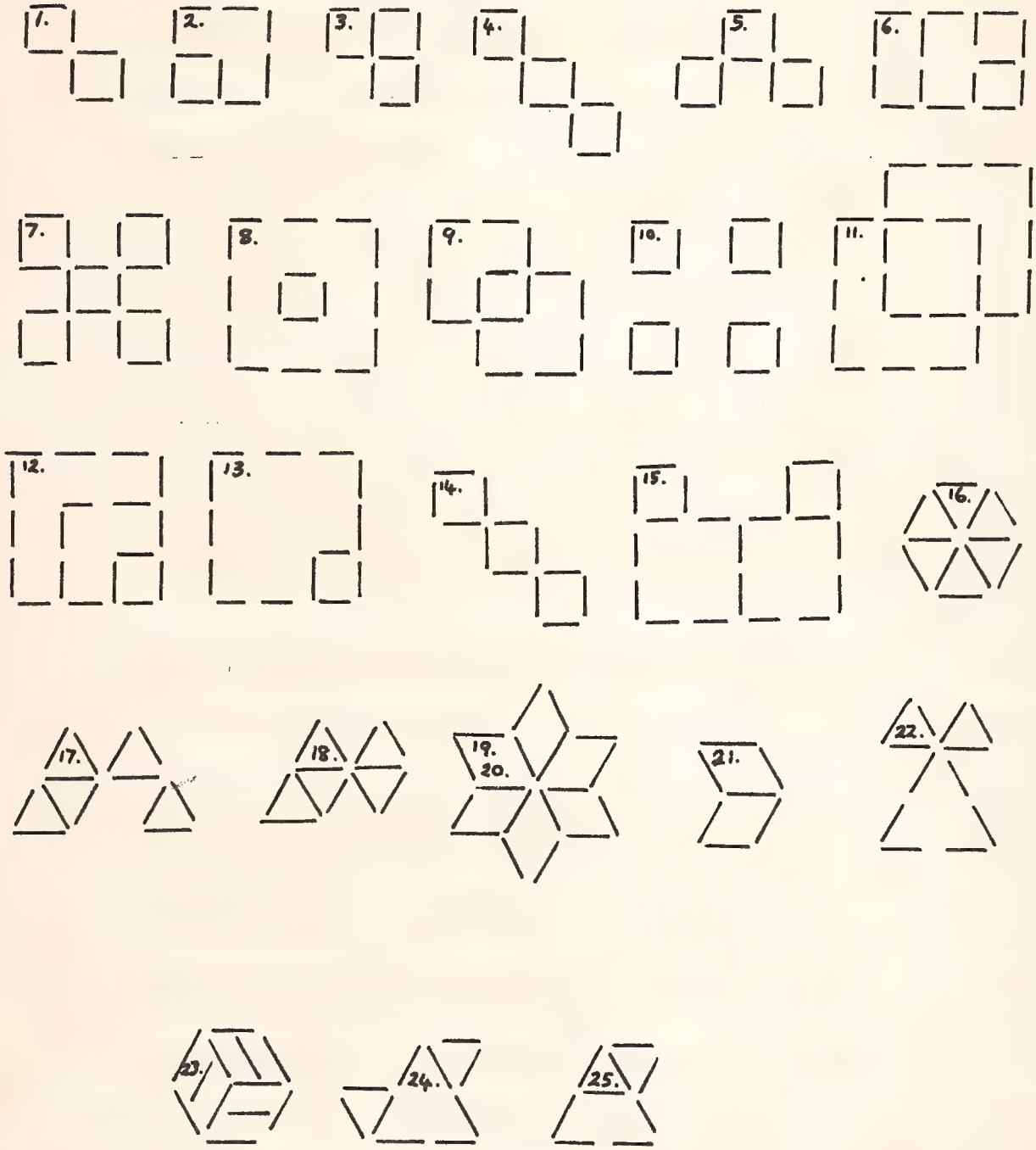
Using 18 matches, it is possible to form two rectangular figures so that the area of one is twice the area of the other (with no sides in common).

Using 18 matches form 2 quadrilaterals so that the area of one is three times the area of the other.

34. Using 12 matches form a figure having an area of 3 square matchstick units.

35. Form 5 equilateral triangles using 9 matches.
36. Form 4 equilateral triangles using 6 matches.

SOLUTIONS



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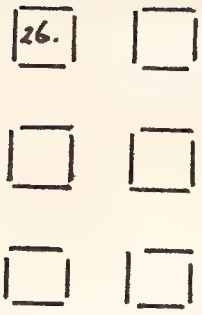

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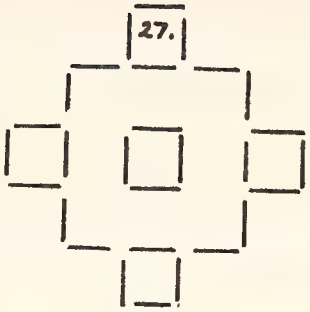

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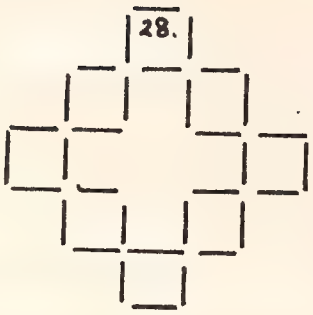

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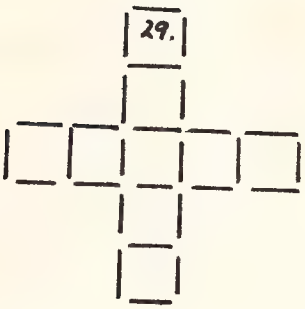

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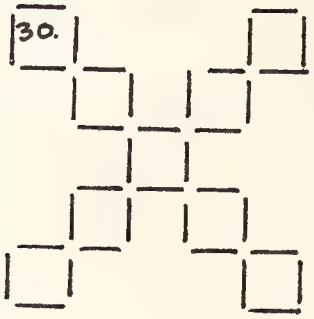

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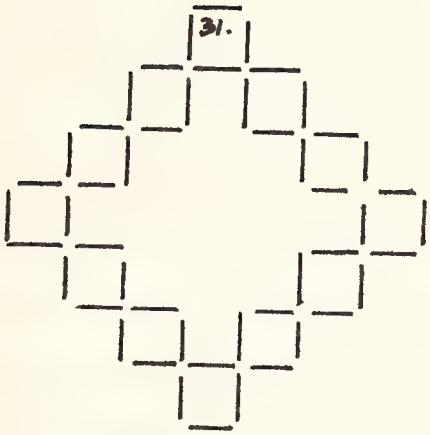

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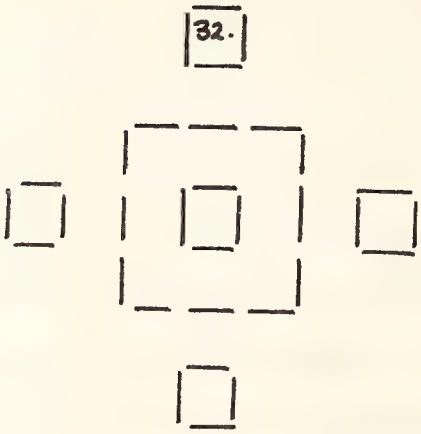

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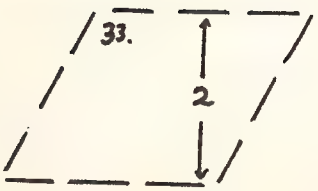

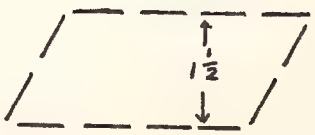

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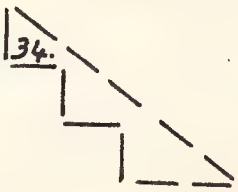


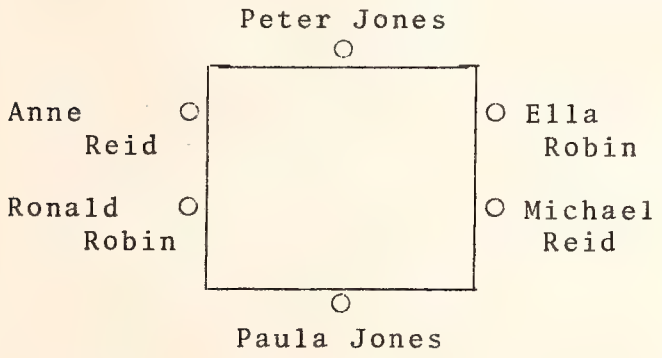
35.

36.

ARRANGEMENTS, DERANGEMENTS, MÉNAGES.

M. Sved.

The Jones' prepare for a dinner-party. Husband and wife are busy setting the table and putting out place cards. "Watch it, Peter, that no husband sits next to his wife."



"Easy! Once you decide where the wives should sit, there is no problem left, there is only one solution. Of course, you may arrange the seating of the wives for a start in 12 different ways, but I think that you agree that the two of us should sit at the ends of the table and that narrows the choice."

"With our small party it is simple, but you should have seen Anne Reid when they bought their big new diningroom table and she prepared for the first of her "Nights at the Round Table." She fussed endlessly with the seating arrangements." "This is hardly surprising. She had to seat 7 couples. After deciding where to place the wives, there are still

$$U_7 = 7! - \frac{14}{13}\binom{13}{1}6! + \frac{14}{12}\binom{12}{2}5! - \frac{14}{11}\binom{11}{3}4! + \frac{14}{10}\binom{10}{4}3! -$$

$$- \frac{14}{9}\binom{9}{5}2! + \frac{14}{8}\binom{8}{6}!1! - \frac{14}{7}\binom{7}{7}0! = 579.$$

ways in which the husbands can be separated from their wives. Since the wives can be seated in 2.7! ways, this gives

$10080 \times 579 = 5{,}836320$ solutions to her problem."

Our friend Peter has of course a sheet of formulae in one hand and an electronic calculator in the other hand when he throws these figures at Paula.

The enumeration of the ways in which n couples can be seated at a circular table so that no husband should sit next to his wife is an old celebrated problem which occupied the minds of distinguished mathematicians in the last century, (Cayley 1878, Lucas 1891). The formula for U_n as shown above was first published by Touchard (1934) and a short proof, which however, requires deep thought was given by Kaplansky (1943). The problem, named "problème des ménages" by Lucas belongs to a class of enumeration problems which can be dealt with by <u>the principle of inclusion and exclusion</u>.

The general logical theorem can be stated as follows: Suppose that a set consists of N objects some of which have one or more of the properties p,q,r,s etc. Denote the numbers of the objects having properties $p,q,r\ldots$ by N_p, N_q, N_r ... respectively, the numbers with joint properties p and q, p and r and so on by N_{pq}, N_{pr}, ... In this notation N_{pqrs} means the numbers of objects with properties p and q and r and s.

The number of objects in the set N which have <u>none of the properties</u> $p,q,r,s,\ldots$ is then given by

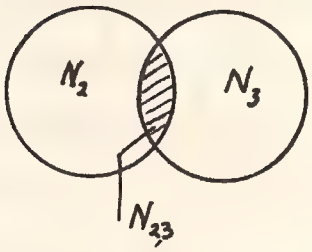
$$N_0 = N-N_p-N_q-N_r-N_s-\ldots+N_{pq}+N_{pr}+\ldots-N_{pqr}-\ldots+N_{pqrs}+\ldots$$

The formula can be proved generally, e.g. by induction on the number of properties, but in preference we illustrate the principles involved in the proof and the applications, by some examples.

<u>Example 1</u> Find the number of all positive integers less than 120 and coprime to 120, i.e. have no divisors other than 1 common with 120.

<u>Solution</u> We must exclude multiples of 2,3,5: prime divisors of 120, each multiple to be excluded exactly once. N_2, N_3, N_5, $N_{2,3}$, $N_{2,5}$, $N_{3,5}$ and $N_{2,3,5}$ respectively represent the multiples of the named primes according to the general notation used in the above formula.

Begin with the multiples of 2 or 3. Their total number is



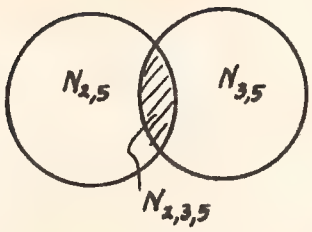
$$N^* = N_2 + N_3 - N_{2,3} \qquad (1)$$

This is clear, if we refer to the Venn diagram. In the sum $N_2 + N_3$ the multiples of 2 and 3, i.e. of 6 are counted twice, hence $N_{2,3}$ must be subtracted.

By the same reasoning the number of multiples of 2,3 or 5 is

$$N_2 + N_3 - N_5^* \qquad (2)$$

where N_5^* denotes the number of those multiples of 5 which were included in the number N^*, (i.e. the numbers divisible by 2 <u>or</u> 3 <u>and</u> 5).



The second Venn diagram shows how to calculate N_5^*:

$$N_5^* = N_{2,5} + N_{3,5} - N_{2,3,5}. \qquad (3)$$

Combine (1), (2) and (3) to obtain

$$(N_2 + N_3 - N_{2,3}) + N_5 - (N_{2,5} + N_{3,5} - N_{2,3,5})$$

for the number of the multiples of 2,3 or 5. Thus the

number of coprimes to 120 is

$$N_0 = N - N_2 - N_3 - N_5 + N_{2,3} + N_{2,5} + N_{3,5} - N_{2,3,5},$$

where $N = 120$, $N_2 = \frac{120}{2} = 60$, $N_3 = \frac{120}{3} = 40$, $N_5 = \frac{120}{24}$

$N_{2,3} = \frac{120}{6} = 20$, $N_{2,5} = \frac{120}{10} = 12$, $N_{3,5} = \frac{120}{15} = 8$,

$N_{2,3,5} = \frac{120}{30} = 4$. Hence $N_0 = 32$.

The problem generalises readily. If the prime-factors of an integer n are $p_1, p_2 \ldots p_r$, then the number of positive integers less than n and coprime to n is

$$\varphi(n) = n - \frac{n}{p_1} - \frac{n}{p_2} - \ldots + \frac{n}{p_1 p_2} + \ldots + (-1)^r \frac{n}{p_1 p_2 \cdot\cdot p_r}$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_r}\right).$$

<u>Example 2</u>. (The absent-minded secretary.) Six letters and six addressed envelopes are put in front of the secretary. She puts <u>each</u> of these letters into the <u>wrong</u> envelope. In how many ways can such a gross mistake be made?

<u>Solution</u>. Denote by N the total number of arrangements,

by N_1 those where the 1st letter is placed correctly

by N_2 the number of correct placements of the second letter, and so on.

Similarly $N_{1,2}$, $N_{1,2,3}$ etc. represent the numbers of correct placements of the named letters.

Then

$$N_0 = N - N_1 - N_2 \ldots - N_6 + N_{1,2} + N_{1,3} + \ldots - N_{1,2,3} - \ldots + N_{1,2,3,4,5,6}.$$

Here $N = 6!$, $N_1 = N_2 = \ldots = N_6 = 5!$ (the total number of arrangements of the remaining letters.)

Similarly $N_{1,2} = \ldots = N_{5,6} = 4!$ and so on.

Thus

$$N_0 = 6!-6.5! + \binom{6}{2}4! - \binom{6}{3}3! + \binom{6}{4}2! - \binom{6}{5}1! + \binom{6}{6}0! = 265.$$

This problem of permutations of n elements such that no element remains in its natural place is called the problem of <u>derangements</u>. The number of derangements of n is

$$D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \ldots + (-1)^n\binom{n}{n}.0!$$

$$\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n\frac{1}{n!}$$

so when $n \to \infty$, $\frac{D_n}{n!} \to e^{-1}$ (e the base of natural logarithms).

For another example on the inclusion-exclusion principle see the solution in this issue of the "multinational problem". (Open section 011).

<u>References</u>: <u>H.J. Ryser</u>. Combinatorial Mathematics. Chapter 2.

<u>I. Kaplansky</u>. Solution of the "Problème des ménages," Bulletin Amer. Math. Soc. Vol.49 (1943), pp.784-785.

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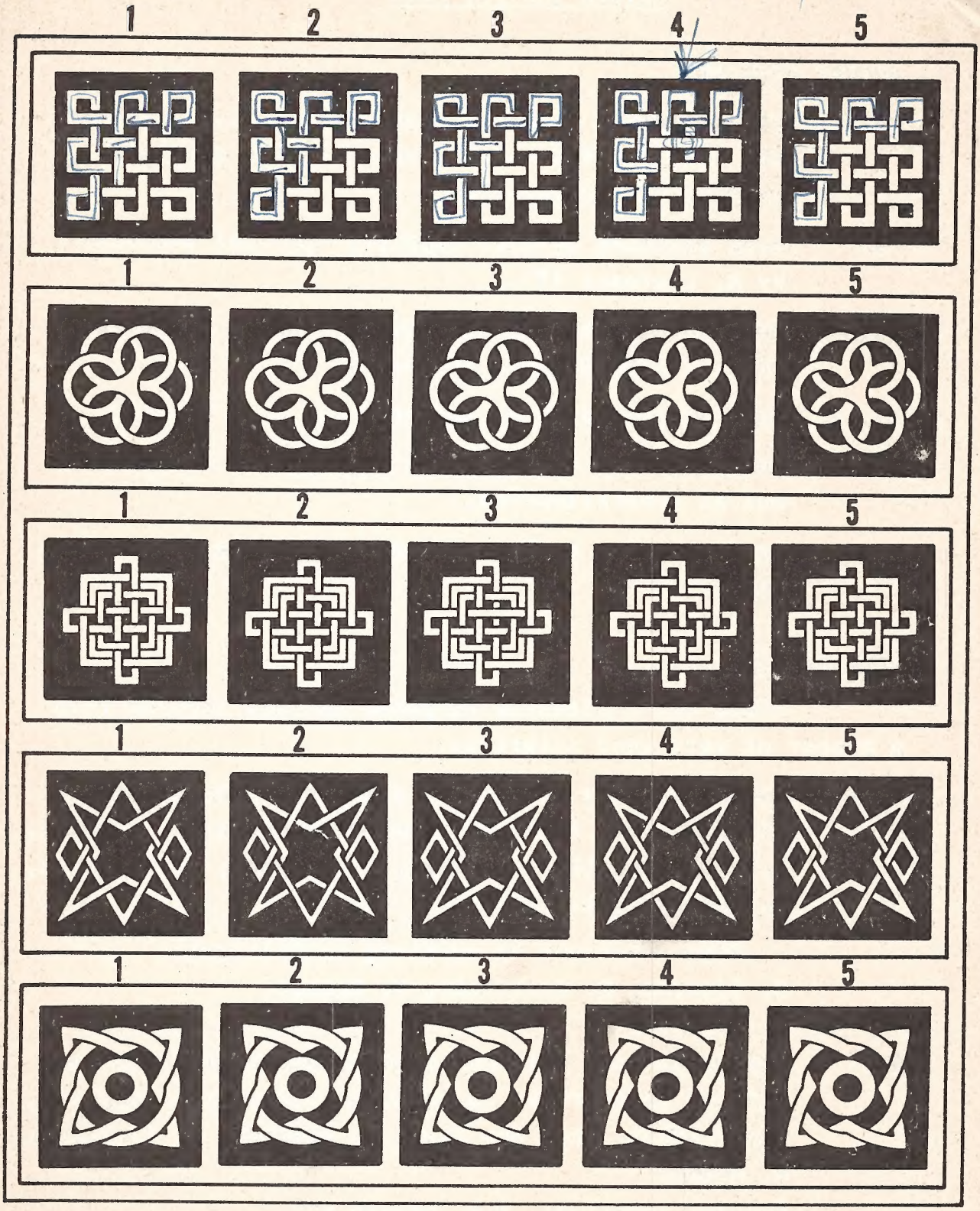
WHICH ONE DIFFERS ?



REPRINTED FROM AFTERMATH 3

SOLUTIONS TO PROBLEMS IN TRIGON VOL.13, No.2

Junior Problems

8. Good solution by H.J. Seppelt (S.P.S.C., 3rd year) who also pointed out the error in the original wording of the question.

9. Good solution by H.J. Seppelt; partial solution by M. Lawrence (*).

10. Good solutions by H.J. Seppelt, and M. Lawrence (Westminster School, 3rd year).

11,13. Good solutions by M. Lawrence and H.J. Seppelt.

12. Good solution to the first part by H.J. Seppelt, but no generalisation.

14. Good solution by H.J. Seppelt, with two alternative forms.

Open Section

There was a pleasing response to this section with a number of very good solutions.

0.8. Good solutions by methods similar to the printed solution were received from Mark Yudkin, A. Capozza, B.R. Dansie, Chia Mon Ming, J. Fargher, L. Heitmann, M. Lawrence, D. Phillis.

Good solutions by equating coefficients: G. Edwards, R. Groblicki, M. Webber.

0.9. Good solutions (without however accounting for the case of equal roots in both equations) by A. Capozza, Mark Yudkin, Chia Mon Ming, G. Edwards, R. Groblicki, M. Webber.

Partial solution: B.R. Dansie (*).

010. Good solutions by A. Capozza, Chia Mon Ming, B.R. Dansie, R. Groblicki, H.J. Seppelt, Mark Yudkin.

Partial solutions: G. Edwards, D. Phillis M. Webber. B.R. Dansie and M. Webber gave neat graphic l illustrations of the problem.

011. There were several good efforts, but no solution succeeded in counting choices without overlaps. (No wonder!)

012. Good solutions by Chia Mon Ming, G. Edwards, R. Groblicki, M. Webber, M. Yudkin.
Partial solution by B.R. Dansie.

013. Partial solutions by Chia Mon Ming, B.R. Dansie, M. Yudkin, G. Edwards, R. Groblicki.

Note: In all the solutions it was assumed that the sum of irrational numbers is irrational. This must be proved in the given case. For example, $3 + \sqrt{5}$, $3 - \sqrt{5}$ are irrational, but their sum is rational.

014. Good solutions by Chia Mon Ming, G. Edwards, M. Yudkin, M. Webber.

(*) A logical error often occurs in solutions to problems where some identity A is to be established. The solutions are set out as follows:
(i) $A \Rightarrow B$ (ii) B is correct (iii) $\therefore$ A is correct.
We illustrate the error in this type of reasoning.
Let A denote "1 = 2".
Hence 2 = 1, so adding we obtain B: "3 = 3".
B is true; $\therefore$ A is true.

Obvious nonsense! If the truth of B can be established, we must check that $B \Rightarrow A$ in order to establish the truth of A.